Backpaper Exam - Optimization
B. Math III

03 June, 2026

- (i) Duration of the exam is 3 hours.
- (ii) The maximum number of points you can score in the exam is 100 (total = 110).
- (iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (10 points) Let $C \subset \mathbb{R}^n$ be closed and convex, and P_C denote the metric projection onto C. Let $p = P_C(x)$ for some $x \notin C$. Prove that

$$\langle x - p, y - p \rangle \leq 0 \quad \text{for all } y \in C.$$

Total for Question 1: 10

2. For each of the following functions, determine whether it is convex, concave or neither (with proper justification):
   (a) (3 points) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{>0}$;
   (b) (3 points) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{>0}$;
   (c) (4 points) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{>0}$.

Total for Question 2: 10

3. Consider an LP in standard form with cost vector $\mathbf{c}$. Let $\mathbf{x}$ be a basic feasible solution associated with the basis matrix $\mathbf{B}$ and $\bar{\mathbf{c}}$ be the corresponding reduced cost vector.

   (a) (10 points) If $\bar{\mathbf{c}} \geq \mathbf{0}$, show that $\mathbf{x}$ is optimal.

   (b) (10 points) If $\mathbf{x}$ is optimal and nondegenerate, then show that $\bar{\mathbf{c}} \geq \mathbf{0}$.

Total for Question 3: 20

4. Consider the following partially completed simplex tableau for a **minimization** problem in standard form. The entries marked with letters are unknown parameters.

	x_1	x_2	x_3	x_4	x_5	x_6
$-d$	0	a	0	b	0	c
4	1	-1	0	2	0	e
2	0	f	1	g	0	-2
3	0	3	0	-1	1	h

State the most general conditions on the unknowns a, b, c, d, e, f, g, h such that:

   (a) (4 points) The current basis is feasible.

   (b) (4 points) The current basic feasible solution is optimal.

   (c) (4 points) The optimal cost is $-\infty$ (the problem is unbounded).

   (d) (4 points) The current basic feasible solution is degenerate.

   (e) (4 points) There exists an alternative optimal solution (assuming the current basis is optimal).

Total for Question 4: 20

5. A furniture shop produces two types of items: **Tables** (x_1) and **Chairs** (x_2). The profit for each table is ₹300 and for each chair is ₹200. The production is limited by two resources:

   - **Labor:** Each table requires 2 hours, and each chair requires 1 hour. Total available labor is 100 hours.
   - **Wood:** Each table requires 1 unit, and each chair requires 1 unit. Total available wood is 80 units.

   (a) (10 points) Suppose the optimal production plan is $x_1^* = 20$ and $x_2^* = 60$. Using **Complementary Slackness**, determine the optimal shadow prices or dual variables (y_1^*, y_2^*).

   (b) (10 points) Should the shop owner pay ₹150 for an additional unit of wood? Should they pay ₹80 for an additional hour of labor? Justify your answers.

Total for Question 5: 20

6. (10 points) Check the optimality of the proposed solution using complementary slackness:

$$\text{maximize} \;\; 4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$$

subject to

$$
\begin{aligned}
x_1 - 4x_3 + 3x_4 + x_5 + x_6 &\le 10 \\
5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &\le 4 \\
4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &\le 4 \\
-x_2 + 2x_4 + x_5 - 5x_6 &\le 6 \\
-2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &\le 12 \\
2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &\le 16 \\
x_1, \ldots, x_6 &\ge 0
\end{aligned}
$$

Proposed solution: $x_1 = 0, x_2 = 0, x_3 = \frac{2}{3}, x_4 = \frac{5}{2}, x_5 = \frac{7}{2}, x_6 = \frac{1}{2}$.

Total for Question 6: 10

7. (20 points) Consider the convex optimization problem:

$$
\begin{array}{rl}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\
 & Ax = b
\end{array}
$$

where $f_0, \ldots, f_m$ are convex functions and $\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i$ represents the problem domain. Suppose that Slater's condition holds for this problem; that is, there exists a point $x \in \operatorname{relint} \mathcal{D}$ such that $f_i(x) < 0$ for all $i = 1, \ldots, m$ and $Ax = b$. Prove that if the primal optimal value p^* is finite, then strong duality holds ($d^* = p^*$) and there exists a dual optimal solution (λ^*, ν^*).

Total for Question 7: 20